WHAT IS...

a Syzygy?

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Linear algebra over rings is lots more fun than over fields. The main reason is that most modules over a ring do not have bases—that is, their generators usually satisfy some nontrivial relations or "syzygies". Given a finitely generated R-module M (where R is a commutative ring) and a set $z_1, \ldots, z_n$ of generators, a *syzygy* of M is an element $(a_1, \ldots, a_n) \in R^n$ for which $a_1 z_1 + \cdots + a_n z_n = 0$. The set of all syzygies (relative to the given generating set) is a submodule of R^n, called the *module of syzygies*. Thus the module of syzygies of M is the kernel of the map $R^n \xrightarrow{\epsilon} M$ that takes the standard basis elements of R^n to the given set of generators.

Suppose, for example, that $R = \mathbb{C}[x, y]$ and $M = R/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal consisting of polynomials with no constant term. The module of syzygies is $\mathfrak{m}$, which is generated by x and y. These elements are not independent, since they satisfy the non-trivial relation $(-y)x + xy = 0$. So we look at the module S of syzygies of $\mathfrak{m}$, which is the rank-one free module generated by the element $(-y, x) \in R^2$. Thus after taking syzygies *twice* we have "resolved" M and obtained a free module. We say that S is the *second* module of syzygies of $R/\mathfrak{m}$. We can encode this information in the following exact sequence (where we have written elements of R^2 as columns, so that matrices act on the left):

$$0 \to R \xrightarrow{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]} R^2 \xrightarrow{[x\ y]} R \xrightarrow{\epsilon} R/\mathfrak{m} \to 0$$

For a slightly more complicated example, let $R = \mathbb{C}[x, y, z]$ and resolve the module $M := R/(x^3y + z^4, xy^2, x^3 + y^3, z^5)$. Using the software package MACAULAY 2, one obtains a free resolution of the following form (the rather complicated matrices defining the maps between free modules need not concern us):

$$0 \to R^3 \xrightarrow{\partial_3} R^6 \xrightarrow{\partial_2} R^4 \xrightarrow{\partial_1} R \xrightarrow{\epsilon} M \to 0$$

This time it takes three steps to resolve the module: The first module of syzygies of M is $\ker(\varepsilon)$, the second (i.e., the module of syzygies of $\ker(\varepsilon)$) is $\ker(\partial_1)$, and the third module of syzygies is $\ker(\partial_2)$. The 0 at the left end of the exact sequence then tells us that ∂_3 maps R^3 isomorphically onto $\ker(\partial_2)$, that is, the third syzygy of M is free.

Hilbert's famous "Syzygy Theorem", published in 1890, states that every finitely generated *graded* module $M = \oplus_{i=0}^{\infty} M_i$ over the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ has a free resolution of length at most n, that is, its n^{th} syzygy is free. (The grading respects the action of the variables, in the sense that $x_j M_i \subseteq M_{i+1}$ for all i and all $j \le n$. The *length* is one less than the number of free modules in the resolution.) Hilbert used this result to prove that the function $i \mapsto \dim_{\mathbb{C}} M_i$ is, for large i, a polynomial function of i. The idea here is that many properties of modules are easy to compute for free modules and behave well along exact sequences. Thus, if a module has a finite free resolution, one can easily read off certain kinds of numerical data.

It turns out that *every* finitely generated module M (not necessarily graded) over a polynomial ring $R = k[x_1, \ldots, x_n]$, where k is a field, has a free resolution of length at most n. By introducing another variable and "homogenizing", one can obtain a free resolution of length at most $n + 1$. Using homological ideas available in the 1950s, one can then show that the n^{th} syzygy S of M is *stably* free, that is, $S \oplus R^p \cong R^q$ for suitable integers p, q. Now one uses "Serre's Conjecture", proved independently by Quillen and Suslin in 1976: projective

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modules over polynomial rings (such as the R-module S above) are free.

Let's switch gears, from graded rings to local rings. Suppose R is a (commutative, Noetherian) local ring with maximal ideal $\mathfrak{m}$, and let M be a finitely generated R-module. By choosing a minimal generating set for M, and then a minimal generating set for the first syzygy, and so on, one obtains a free resolution

$$\cdots \to R^{b_n} \to \cdots \to R^{b_1} \to R^{b_0} \to M \to 0.$$

It is not hard to see that the syzygies are uniquely determined up to isomorphism (independent of the choice of generators at each stage). We let $\mathrm{syz}^i(M)$ denote the i^{th} syzygy, that is, the image of the map $R^{b_i} \to R^{b_{i-1}}$. The integers $b_i = b_i(M)$ are called the *Betti numbers* of M. If, for some h, we have $b_h(M) \neq 0$ but $b_i(M) = 0$ for $i > h$, we say M has *projective dimension* h. For the ring of formal power series $\mathbb{C}[[x_1, \ldots, x_n]]$, it follows from Hilbert's theorem that every finitely generated module M has projective dimension at most n. On the other hand, for the cuspidal ring $R := \mathbb{C}[[t^2, t^3]] = \mathbb{C}[[x, y]]/(y^2 - x^3)$, the minimal resolution of the simple module $R/(x, y)$ is easily seen to be infinite, and periodic after one step:

$$\cdots \to R^2 \xrightarrow{\begin{bmatrix} y & -x^2 \\ -x & y \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} y & x^2 \\ x & y \end{bmatrix}} R^2$$
$$\xrightarrow{\begin{bmatrix} y & -x^2 \\ -x & y \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} R \to M \to 0$$

On the other hand, the Betti numbers of the module $\mathbb{C}[[t^3, t^4, t^5]]/(t^3, t^4, t^5)$ over the ring $\mathbb{C}[[t^3, t^4, t^5]]$ grow exponentially.

These examples are symptoms of general behavior promised by deep results in the representation theory of local rings. First and foremost is the theorem of Auslander, Buchsbaum, and Serre, which says that a local ring R is regular (meaning, in the geometric context, that it corresponds to a smooth point on a variety) if and only if every finitely generated R-module has finite projective dimension. Next, if R is any ring of the form $\mathbb{C}[[x_1, \ldots, x_n]]/(f)$, where f is a nonzero element in the square of the maximal ideal, then the minimal resolution of every finitely generated R-module is eventually periodic, of period at most two. To place the last example in context, see Avramov's survey [1] of the asymptotic behavior of Betti numbers over local rings.

Syzygies and the structure of free resolutions are currently the subject of intense research, though I have mentioned only a few highlights. For an account of the use of syzygies in algebraic geometry, cf. Eisenbud's book [2]. To see why applied mathematicians should know about syzygies, cf. David Cox's article in the November 2005 *Notices*.

I will close by mentioning a few open problems. First, as might be suggested by our first example, for the simple module $R/\mathfrak{m}$ over a regular local ring R of dimension n, the Betti numbers are binomial coefficients: $b_i = \binom{n}{i}$. The "Horrocks conjecture" asserts that these are the *smallest possible* Betti numbers for a module of projective dimension n. A related problem is the "syzygy conjecture", which asserts that if M has projective dimension n then $b_i - b_{i+1} + b_{i+2} - \cdots \pm b_n \geq i$ for each $i < n$. The conjecture is true for rings containing a field [3], but it is still open in the general case. Another open question concerns modules of infinite projective dimension: Given a finitely generated module M over an arbitrary local ring R, are the Betti numbers $b_i(M)$ eventually nondecreasing?

Further Reading

[1] L. Avramov, Homological asymptotics of modules over local rings, *Commutative Algebra* (M. Hochster, C. Huneke, J. D. Sally, eds.), Mathematical Sciences Research Institute Publications, vol. 15, 1987.

[2] D. Eisenbud, *The Geometry of Syzygies*, Graduate Texts in Mathematics, vol. 229, Springer, 2005.

[3] E. G. Evans and P. Griffith, *Syzygies*, London Mathematics Society Lecture Notes Series, vol. 106, 1985.