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About the cover: Rational curves on a K3 surface

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The picture shows (part of the real locus of) a sextic plane curve

$$C : S(x_0, x_1, x_2) = 0$$

singular only at one simple node, together with numerous lines tritangent to C and several other rational plane curves on which S is a perfect square. These geometrical configurations yield infinite families of curves of genus 2 over $\mathbf{Q}$ with many rational points.

The sextic arises from the K3 surface $\mathcal{X}$ obtained by resolving the node of the double cover of the projective plane $\mathbf{P}^2$ branched on C. This $\mathcal{X}$ is a "singular" K3 surface: its Néron–Severi group $\mathrm{NS}(\mathcal{X})$ has rank 20, the largest possible for a K3 surface in characteristic zero. Moreover, $\mathcal{X}$ has a model over $\mathbf{Q}$ for which $\mathrm{NS}(\mathcal{X})$ consists entirely of classes of divisors defined over $\mathbf{Q}$. We show (as does M. Schütt [**Sch08**, Theorem 1] by a different argument) that a singular K3 surface X with this property must have $\mathrm{disc}(\mathrm{NS}(X)) \in \mathbf{\Delta}_1$, where

$$\mathbf{\Delta}_1 = \{-3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -163\}$$

is the set of 13 discriminants of imaginary quadratic orders with unique factorization; all these discriminants occur, each for a unique X except that the surfaces with discriminants -3 and -4 are subject to cubic and quadratic twist, respectively. Discriminant -163 gives rise to our $\mathcal{X}$.

We represent $\mathcal{X}$ as a double cover of $\mathbf{P}^2$ using the 3-dimensional space $\Gamma(D)$ of sections of a numerically effective divisor D such that $D \cdot D = 2$. In general the branch locus of the double cover $\mathcal{X} \to \mathbf{P}^2$ is a sextic curve that may be singular or even reducible. The singularities are the images of smooth rational curves r on $\mathcal{X}$ such that $r \cdot D = 0$. Thus they can be described as follows in terms of $\mathrm{NS}(\mathcal{X})$. Let L be the sublattice of $\mathrm{NS}(\mathcal{X})$ orthogonal to D, made positive definite by multiplying all the norms by -1. Then L is an even lattice, and has a canonical sublattice, the *root lattice* R_L spanned by vectors of norm 2 (the *roots* of L), which decomposes uniquely as the direct sum of simple root lattices A_n, D_n, E_n. Each of the simple root lattices in R_L comes from a set of rational curves r mapping to a singularity of the corresponding type of the branch locus of the double cover.

Roughly speaking, the larger $\mathrm{disc}\, L$ is allowed to be, the smaller R_L can be made. In our case $\mathrm{disc}(L) = 2 \cdot 163$. We cannot quite make R_L trivial, but there are several choices that make $R_L \cong A_1$, giving a sextic curve with a single node.

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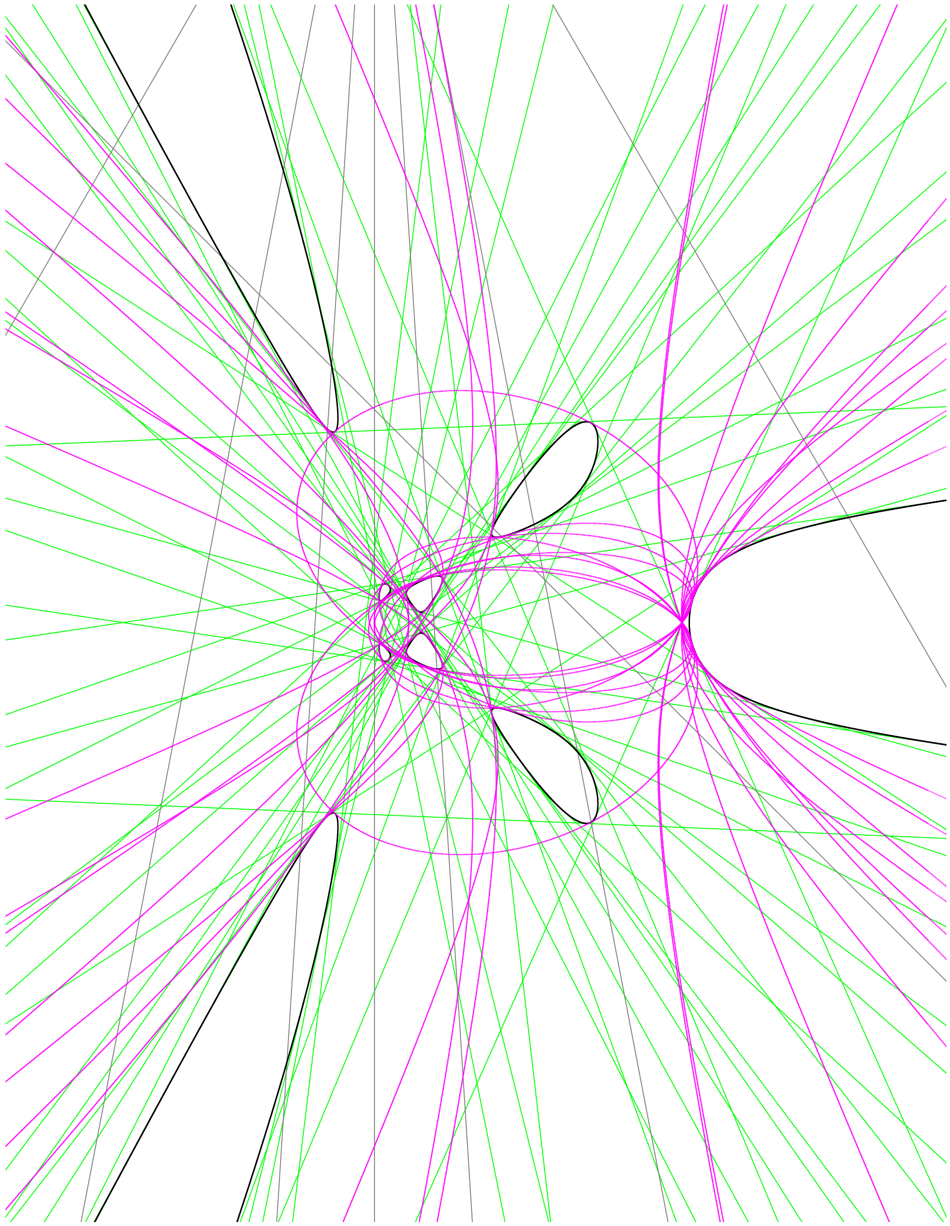

FIGURE 1. Branch locus and rational curves on the K3 surface $\mathcal{X}$

Our picture shows one of these, which is the only such choice for which L has a symmetry other than ± 1; this symmetry is inherited by the sextic curve. Explicitly, we have a basis (x_0, x_1, x_2) for $\Gamma(D)$ for which S can be given by

$$\begin{aligned} S(x_0,x_1,x_2) \;=\;& 3686400x_0^6 \\ & - 256(33975x_1^2 - 8569x_1x_2 - 45x_2^2)x_0^4 \\ & + (5130225x_1^4 - 1860100x_1^3x_2 + 138414x_1^2x_2^2 - 4180x_1x_2^3 + 9x_2^4)x_0^2 \\ & - 8(5643x_1^5x_2 - 2495x_1^4x_2^2 + 209x_1^3x_2^3 - 5x_1^2x_2^4). \end{aligned}$$

The symmetry takes $(x_0 : x_1 : x_2)$ to $(-x_0 : x_1 : x_2)$, and the node is at $(x_0 : x_1 : x_2) = (0 : 0 : 1)$. Our model of $\mathcal{X}/\mathbf{Q}$ is obtained from the double cover $y^2 = S(x_0, x_1, x_2)$ by blowing up the preimage of this node. In the picture, the real locus of the sextic curve $C : S = 0$ is plotted in black in the $(x_0/x_2, x_1/x_2)$ plane; it consists of nine components, together with an isolated point at the node.

A line $\ell \subset \mathbf{P}^2$ is tritangent to C if and only if it lifts to a pair of smooth rational curves $\ell_\pm$ on $\mathcal{X}$. Then $\ell_+ + \ell_- = D$ in $\mathrm{NS}(\mathcal{X})$. Orthogonal projection to $L \otimes \mathbf{Q}$ then maps $\mathrm{NS}(\mathcal{X})$ to a lattice L' containing L with index 2, taking the curves $\ell_\pm$ to a pair of vectors $\pm v = \ell_\pm - \frac{1}{2}D \in L'$ of norm 5/2 that are orthogonal to R_L. Conversely, every such pair comes from a tritangent line. There are 43 such lines; one of these is $x_2 = 0$, which is the line at infinity in our picture, and the remaining 42 are plotted in green. (Some of the tangency points are not in the picture because they are either complex conjugate or real but outside the picture frame.)

Each of these lines has the property that the restriction $S|_\ell$ is the square of a cubic polynomial. The same is true if ℓ is a line passing through the node of C and tangent to C at two other points. There are nine such lines, plotted in gray. They correspond to norm-(5/2) vectors in L' not orthogonal to R_L, up to multiplication by -1 and translation by R_L.

A generic line $\lambda \subset \mathbf{P}^2$ meets these $43+9$ lines in 52 distinct points that lift to 52 pairs of rational points on the genus-2 curve $y^2 = S|_\lambda$. This already improves on the previous record for an infinite family of genus-2 curves over $\mathbf{Q}$ (which was 24 pairs, due to Mestre). We do better yet by exploiting rational curves of higher degree in $\mathbf{P}^2$ on which S restricts to a perfect square.

There are 1240 conics $c \subset \mathbf{P}^2$ for which $S|_c$ is a square; geometrically these are the conics such that each point in the intersection $c \cap C$ has even multiplicity (either the node of C or a point of tangency). Such a conic lifts to a pair of rational curves $c_\pm$ on $\mathcal{X}$ with $c_+ + c_- = 2D$. These $c_\pm$ come from vectors $c_\pm - D \in L$ of norm 4 up to translation by R_L, except for norm-4 vectors of the form $v - v'$ with $v, v' \in L'$ of norm 5/2. The conics c are all rational over $\mathbf{Q}$, because for each c we can find c' such that $c_+ \cdot c'_+$ is odd. In general the intersections of c with a generic line $\lambda \subset \mathbf{P}^2$ need not be rational, but we can choose λ so as to gain a few rational points. Most notably, 18 of the conics happen to pass through the point $P_0 : (x_0 : x_1 : x_2) = (0 : 1 : 3)$ on the axis of symmetry $x_0 = 0$ of the sextic C. These conics are plotted in purple on our picture. If λ is a generic line through P_0 then the genus-2 curve $y^2 = S|_\lambda$ gains 18 more pairs of points above the second intersections of λ with the purple conics. We also lose one pair because two of our 52 tritangent lines pass through P_0, but we gain two more pairs by finding two rational cubic curves $\kappa \subset \mathbf{P}^2$ for which $S|_\kappa$ is a square and P_0 is the node of κ. This brings the total to $52+18-1+2 = 71$. If c_1, c_2 are two of the remaining 1222 conics such that $(c_1)_+ \cdot (c_2)_+$ is odd then we have infinitely many choices (parametrized by an elliptic curve of positive rank) of lines $\lambda \ni P_0$ for which each of $\lambda \cap c_1$ and $\lambda \cap c_2$ consists of two further rational points, bringing our total to 75. This is the current record for the number of pairs of rational points on an infinite family of genus-2 curves over $\mathbf{Q}$; the previous record, due to Mestre, was 24 pairs.

In another direction, C has the rational point $P_1 : (x_0 : x_1 : x_2) = (0 : 1 : 0)$ (in our picture this is the point at infinity in the horizontal direction). If $\lambda \ni P_1$ then the genus-2 curve $y^2 = S|_\lambda$ has a rational Weierstrass point mapping to P_1. The tangent to P_1 is the line of infinity, which is one of our 52 tritangent lines; but this still leaves 51 pairs of rational points. In fact we get 4 more because four of our 1240 conics contain P_1. These are shown in our picture as red horizontal parabolas. As before we can get at least 4 more pairs for infinitely many choices of λ parametrized by an elliptic curve of positive rank. This yields infinitely many genus-2 curves over $\mathbf{Q}$ with a rational Weierstrass point and at least 59 further pairs of rational points.

References

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