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A Note on the Circular Complex Centered Form

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Abstract — Zusammenfassung

A Note on the Circular Complex Centered Form. The centered form for real rational functions suggested by R. E. Moore [6] was extended to complex polynomials over circular complex domains in [7]. Here it is shown that the inclusion chain

circular complex centered form evaluation $\subseteq$
Horner scheme $\subseteq$
power sum evaluation

is valid for all complex polynomials and all circular domains.

Eine Bemerkung über die zentrierte Form über komplexen Kreisintervallen. Für reelle rationale Funktionen wurde die zentrierte Form von R. E. Moore [6] angedeutet. Diese Form wurde in [7] auch für komplexe Polynome definiert. Wir zeigen, daß die Inklusionskette

zentrierte Kreisintervallform $\subseteq$
Hornerentwicklung $\subseteq$
Potenzreihenentwicklung

für alle komplexen Polynome über Kreisintervallen gültig ist.

0. Introduction

In a previous paper [7] we showed that the circular complex centered form was always included in the power-sum evaluation of a complex polynomial over a circular complex interval. In this note we will show that the following ranking of evaluations is valid

circular complex centered form $\subseteq$ Horner scheme evaluation $\subseteq$ power-sum evaluation.

1. Interval Analysis

Interval analysis is now a well established topic in numerical analysis. Definitions of interval arithmetic as well as some theoretical results can be found in [1] and [5].

The advantage of interval arithmetic is that it combines the approximation and the error analysis in one process. For example, it we evaluate the polynomial

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$p(x)=\sum_{i=0}^{n} a_i x^i$ over a real interval $[u,v]$, substituting a_i by $[a_i,a_i]$, x by $[u,v]$ and performing interval arithmetic we will arrive at a final interval $p(X)$. Choosing the center of $p(X)$ as an approximation the radius of $p(X)$ then becomes the error bound.

The notation $I(R)$ will be used to denote the set of all real intervals $[a,b]$, $a,b\in R$. Since a real number a can be written as $[a,a]$ we may regard the real number field as a subset of the set of real intervals $I(R)$.

Let $\mathbb{C}$ be the set of complex numbers, let $c\in\mathbb{C}$ and let $r>0$. The set

$$Z=\{z \mid |z-c|\le r,\ z\in\mathbb{C}\}$$

is then called a complex interval or circular interval. A circular interval may also be written as

$$Z=\langle c,r\rangle$$

and the set of all circular intervals will be denoted by $c(\mathbb{C})$.

Arithmetic operations on $c(\mathbb{C})$ were introduced by Garagantini and Henrici [3]. They defined the operations in the following way:

Let

$$Z_i=\langle c_i,r_i\rangle\in c(\mathbb{C}),\ i=1,\dots,n \text{ and } a\in\mathbb{C}$$

then

$$a+Z=\langle a+c,r\rangle, \tag{1}$$

$$aZ=\langle ac,\,|a|r\rangle, \tag{2}$$

$$\sum_{i=1}^{n} Z_i=\left\langle \sum_{i=1}^{n} c_i,\ \sum_{i=1}^{n} r_i\right\rangle, \tag{3}$$

$$Z_1 Z_2=\langle c_1 c_2,\ |c_1|r_2+|c_2|r_1+r_1 r_2\rangle. \tag{4}$$

Generally the (4) does not result in equality except in special cases. It is only true that

$$\{z_1 z_2 \mid z_1\in Z_1, z_2\in Z_2\}\subseteq Z_1\cdot Z_2.$$

Hence (4) only gives an estimate of the range ([3]). The commutative and the associative law hold for elements from $c(\mathbb{C})$. However, as in the real case we only have the subdistributivity [3]. That is

$$Z_1(Z_2+Z_3)\subseteq Z_1 Z_2+Z_1 Z_3. \tag{5}$$

Other definitions of the multiplication (4) are possible. They are not used here.

2. Estimates for the Range of a Complex Polynomial

Let $p(z)=\sum_{i=0}^{n} a_i z^i$ be a complex polynomial to be evaluated over the circular interval $Z=\langle c,r\rangle$. Substituting a_i by $\langle a_i,0\rangle$, z by $Z=\langle c,r\rangle$ computing Z^i, and performing circular arithmetic operations (1)–(4), we arrive at a final disk $Z_p=\langle c_p,r_p\rangle$ which is an estimation of the range $\bar{p}(Z)=\{p(z)\,|\,z\in Z\}$. We will call this the power-sum method.

From (5) it is easy to see that the Horner scheme might generate a better approximation. According to this scheme we substitute a_i by $A_i=\langle a_i, 0\rangle$, $i=1,\ldots,n$, z by $Z=\langle c, r\rangle$ and proceed as follows:

$$\begin{aligned} &X:=A_n \\ &X:=X*Z+A_i,\ \ i=n-1,\, n-2,\ldots,0 \end{aligned}$$

using circular arithmetic.

The result of the Horner scheme can also be explicitly calculated. This is the content of the following theorem.

Theorem 1:

Let the complex polynomial $p(z)=a_0+a_1z+\ldots+a_nz^n$ be given. Using the Horner scheme to evaluate the polynomial over the circular disk $Z=\langle c,r\rangle$ results in the circular disk $Z_h=\langle c_h, r_h\rangle$ where

$$\begin{aligned} c_h&=p(c) \\ r_h&=r\sum_{k=1}^{n}\left\{(|c|+r)^{k-1}\left|\sum_{j=k}^{n}a_jc^{j-k}\right|\right\}. \end{aligned}$$

Proof:

We first define the following notations.

$$\begin{aligned} p_1(z)&=a_0+z(a_1) \\ p_2(z)&=a_0+z(a_1+z(a_2)) \\ &\ \vdots \\ p_n(z)&=a_0+z(a_1+z(a_2+\ldots+z(a_n)\ldots) \end{aligned}$$

and

$$\begin{aligned} p_1^{(1)}(z)&=a_1+z(a_2) \\ p_2^{(1)}(z)&=a_1+z(a_2+z(a_3)) \\ &\ \vdots \\ p_{n-1}^{(1)}(z)&=a_1+z(a_2+z(a_3+\ldots+z(a_n)\ldots)) \end{aligned}$$

Using this notation it follows that

$$\begin{aligned} p_n(z)&=a_0+z(a_1+z(a_2+\ldots+z(a_n)\ldots)) \\ &=a_0+z\,p_{n-1}^{(1)}(z). \end{aligned}$$

We now use complete induction on n to show that

$$p_n(Z)=\left\langle p_n(c),\ r\sum_{k=1}^{n}\left\{(|c|+r)^{k-1}\left|\sum_{j=k}^{n}a_jc^{j-k}\right|\right\}\right\rangle.$$

For $n=1$, we have

$$\begin{aligned} p_1(Z) &= a_0 + a_1 Z \\ &= a_0 + a_1 \langle c, r \rangle \\ &= \langle a_0 + a_1 c, |a_1| r \rangle \\ &= \left\langle p_1(c),\ r \sum_{k=1}^{1} \left\{ (|c|+r)^{k-1} \left| \sum_{j=k}^{1} a_j c^{j-k} \right| \right\} \right\rangle. \end{aligned}$$

Assume now that

$$p_{n-1}(Z) = \left\langle p_{n-1}(c),\ r \sum_{k=1}^{n-1} \left\{ (|c|+r)^{k-1} \left| \sum_{j=k}^{n-1} a_j c^{j-k} \right| \right\} \right\rangle$$

is valid. Then for $p_n(Z)$ we have

$$\begin{aligned} p_n(Z) &= a_0 + Z \cdot p_{n-1}^{(1)}(Z) \\ &= a_0 + \langle c, r \rangle \left\langle p_{n-1}^{(1)}(c),\ r \sum_{k=1}^{n-1} \left\{ (|c|+r)^{k-1} \left| \sum_{j=k}^{n-1} a_{j+1} c^{j-k} \right| \right\} \right\rangle \\ &= a_0 + \left\langle c \cdot p_{n-1}^{(1)}(c),\ (|c|+r) \left\{ r \sum_{k=1}^{n-1} \left[(|c|+r)^{k-1} \right. \right. \right. \\ &\qquad \left. \left. \left. \left| \sum_{k=1}^{n-1} a_{j+1} c^{j-k} \right| \right] \right\} + r \cdot |p_{n-1}^{(1)}(c)| \right\rangle \\ &= \left\langle a_0 + c \cdot p_{n-1}^{(1)}(c),\ r \left\{ \sum_{k=1}^{n-1} \left[(|c|+r)^{k} \left| \sum_{j=k}^{n-1} a_{j+1} c^{j-k} \right| \right] \right. \right. \\ &\qquad \left. \left. + \left| \sum_{j=1}^{n-1} a_j c^{j-1} \right| \right\} \right\rangle \\ &= \left\langle a_0 + c \sum_{j=1}^{n} a_j c^{j-1},\ r \left\{ \sum_{k=2}^{n} \left[(|c|+r)^{k-1} \left| \sum_{j=k-1}^{n-1} a_{j+1} c^{j-(k-1)} \right| \right] \right. \right. \\ &\qquad \left. \left. + \left| \sum_{j=1}^{n} a_j c^{j-1} \right| \right\} \right\rangle \\ &= \left\langle p_n(c),\ r \left\{ \sum_{k=2}^{n} \left[(|c|+r)^{k-1} \left| \sum_{j=k}^{n} a_j c^{j-k} \right| \right] + \left| \sum_{j=1}^{n} a_j c^{j-1} \right| \right\} \right\rangle \\ &= \left\langle p_n(c),\ r \sum_{k=1}^{n} \left\{ (|c|+r)^{k-1} \left| \sum_{j=k}^{n} a_j c^{j-k} \right| \right\} \right\rangle \end{aligned}$$

which completes the proof.

In other words, instead of using circular arithmetic operations we can calculate the result of the Horner scheme in terms of c, r, and a_i, $i=0,\dots,n$ using complex arithmetic only. This theorem is very useful when we compare the Horner scheme with the centered form.

The centered form for a complex polynomial over a circular disk has been investigated in [7]. Here we give a brief introduction.

Analogous to the real case, we define the centered form for a complex function over a circular domain $Z=\langle c,r\rangle$ as follows:

Let $p(z)$ be written as

$$p(z)=p(c)+t(z-c)$$

for a $z\in Z$ where the complex function t is defined by

$$f(y)=p(y+c)-p(c).$$

For the case of a complex polynomial it follows that

$$t(y)=y\cdot q(y)$$

where q is some complex polynomial. Hence the centered form for a complex polynomial can be written as

$$p_c(z)=p(c)+t(z-c). \tag{6}$$

One way to write the polynomial $p(z)$ in the centered form is to expand $p(z)$ as a Taylor series around the point c. This is the centered form discussed here. Therefore we have the representation

$$p_c(z)=\sum_{k=0}^{n}\frac{p^{(k)}(c)}{k!}(z-c)^k. \tag{7}$$

Furthermore, the result of the centered form can be explicitly calculated by

$$p_c(Z)=\left\langle p(c),\ \sum_{k=1}^{n}\left|\frac{p^{(k)}(c)}{k!}\right|r^k\right\rangle, \tag{8}$$

which has been proven in [7] and [9].

At this point it should be noted that the three forms discussed above all have the property that the center of the evaluation is the function value at the center of the domain. Furthermore, if $Z_1, Z_2\in c(\mathbb{C})$ such that $Z_1\subseteq Z_2$ and Z_x^i, $i=1,2$; $x\in\{p,h,c\}$ are the evaluations of $p_n(z)$ over Z, respectively Z_2 for each of the methods then $Z_x^1\subseteq Z_x^2$. This follows directly from the definitions (1)–(4) using induction.

The quadratic convergence of the centered form has been shown in [7]. However, no comparison between the centered form and the Horner scheme has been made.

3. Comparison

Based on the result of theorem 1 and equation (8), we can prove the following important theorem.

Theorem 2:

Let the complex polynomial $p(z)=a_0+a_1z+\ldots+a_nz^n$ be evaluated over the circular interval $Z=\langle c,r\rangle$. The result of the evaluation using the Horner scheme is denoted by $Z_h=\langle c_h,r_h\rangle$. The result of the centered form evaluation is $Z_c=\langle c_c,r_c\rangle$. Then, we have

$$Z_c\subseteq Z_h.$$

Proof:
From Theorem 1 and (8), we have

$$Z_c = \left\langle p(c),\ \sum_{k=1}^{n} \frac{p^{(k)}(c)}{k!} r^k \right\rangle$$

and

$$Z_h = \left\langle p(c),\ r \sum_{k=1}^{n} \left\{ (|c|+r)^{k-1} \left| \sum_{j=k}^{n} a_j c^{j-k} \right| \right\} \right\rangle.$$

Hence

$$c_c = c_h = p(c),$$

$$r_c = \sum_{k=1}^{n} \left| \frac{p^{(k)}(c)}{k!} \right| r^k$$

and

$$r_h = r \sum_{k=1}^{n} \left\{ (|c|+r)^{k-1} \left| \sum_{j=k}^{n} a_j c^{j-k} \right| \right\}.$$

The theorem is therefore proven if $r_c \leq r_h$. This is now proven.

$$\begin{aligned}
& r \sum_{k=1}^{n} \left\{ (|c|+r)^{k-1} \left| \sum_{j=k}^{n} a_j c^{j-k} \right| \right\} \\
&= r \sum_{k=1}^{n} \left\{ \left[\sum_{i=1}^{k-1} \binom{k-1}{i} |c|^{k-1-i} r^i \right] \left| \sum_{j=k}^{n} a_j c^{j-k} \right| \right\} \\
&= r \sum_{k=1}^{n} \left\{ \sum_{i=0}^{k-1} \left[\binom{k-1}{i} |c|^{k-1-i} r^i \left| \sum_{j=k}^{n} a_j c^{j-k} \right| \right] \right\} \\
&= r \sum_{k=1}^{n} \sum_{i=0}^{k-1} \left\{ \binom{k-1}{i} |c|^{k-1-i} r^i \left| \sum_{j=k}^{n} a_j c^{j-k} \right| \right\} \\
&= r \sum_{i=0}^{n-1} \sum_{k=i+1}^{n} \left\{ \binom{k-1}{i} |c|^{k-1-i} r^i \left| \sum_{j=k}^{n} a_j c^{j-k} \right| \right\} \\
&= \sum_{i=0}^{n-1} \left\{ r^{i+1} \sum_{k=i+1}^{n} \left[\binom{k-1}{i} \left| \sum_{j=k}^{n} a_j c^{j-(i+1)} \right| \right] \right\} \\
&\geqq \sum_{i=0}^{n-1} \left\{ r^{i+1} \left| \sum_{k=i+1}^{n} \left[\binom{k-1}{i} \sum_{j=k}^{n} a_j c^{j-(i+1)} \right] \right| \right\} \\
&= \sum_{i=0}^{n-1} \left\{ r^{i+1} \left| \sum_{k=i+1}^{n} \sum_{j=k}^{n} \binom{k-1}{i} a_j c^{j-(i+1)} \right| \right\} \\
&= \sum_{i=0}^{n-1} \left\{ r^{i+1} \left| \sum_{j=i+1}^{n} \sum_{k=i+1}^{j} \binom{k-1}{i} a_j c^{j-(i+1)} \right| \right\} \\
&= \sum_{i=0}^{n-1} \left\{ r^{i+1} \left| \sum_{j=i+1}^{n} a_j c^{j-(i+1)} \sum_{k=i+1}^{j} \binom{k-1}{i} \right| \right\}
\end{aligned}$$

$$= \sum_{j=0}^{n-1} \left\{ r^{i+1} \left| \sum_{j=i+1}^{n} \binom{j}{i+1} a_j c^{j-(i+1)} \right| \right\}$$

$$= \sum_{i=0}^{n-1} r^{i+1} \left| \frac{p^{(i+1)}(c)}{(i+1)!} \right|$$

$$= \sum_{i=1}^{n} \left| \frac{p^{(i)}(c)}{i!} \right| r^i .$$

Here the equation

$$\sum_{k=i+1}^{j} \binom{k-1}{i} = \binom{j}{i+1}$$

can be verified using complete induction.

Therefore we can make the assertion

$$Z_c \subseteq Z_h .$$

This shows that the centered form always generates a smaller estimate range than the Horner scheme does. Furthermore, we have the following conclusion.

Theorem 3:

Let the complex polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ *be evaluated over the circular interval* $Z = \langle c, r \rangle$. *Let* $Z_p = \langle c_p, r_p \rangle$ *denote the result of the power-sum evaluation;* $Z_h = \langle c_h, r_h \rangle$ *the result of the Horner scheme; and* $Z_c = \langle c_c, r_c \rangle$ *the result of the centered form. Then, we have*

$$\bar{p}(Z) \subseteq Z_c \subseteq Z_h \subseteq Z_p ,$$

where $\bar{p}(Z)$ *denotes the exact range and the equality*

$$Z_c = Z_h = Z_p$$

holds when $c = 0$.

Proof:

The relation

$$\bar{p}(Z) \subseteq Z_c$$

follows from the inclusion property ([1], [7]).

$$Z_c \subseteq Z_h$$

comes from the previous theorem.

The relation

$$Z_h \subseteq Z_p$$

is based on the subdistributivity.

When $c = 0$, we know that the expression

$$p(z) = \sum_{i=0}^{n} a_i z^i$$

is equivalent to the centered form.

Hence

$$Z_c = Z_h = Z_p.$$

Therefore the final conclusion is that the centered form always generates the including circle of smallest radius.

4. Numerical Examples

In this section, we compare the evaluation of the polynomial $p(z) = \sum_{i=0}^{n} a_i z^i$ using the centered form with the Horner scheme and power-sum method. A graphical estimate of the range is obtained by mapping a sequence of points on the domain circle.

Example 1: We first choose the polynomial

$$\begin{aligned} p(z) &= (.1271 - i\,.9173) + (.9115 - i\,.9381)\,z \\ &= (.9125 - i\,.9821)\,z^2 + (.3541 + i\,.9368)\,z^3 \\ &+ (.4721 + i\,.7631)\,z^4 + (.4925 + i\,.5812)\,z^5 \end{aligned}$$

to be evaluated over the disk $\langle .1024 + i.2013,\ 7514 \rangle$. The numerical results are shown in the following table.

Table 1

Algorithm	Resulting Circle	Area of Resulting Circle
Power Sum	$\langle .4768 - i\,.8510,\ 4.612 \rangle$	66.82
Hornerscheme	$\langle .4768 - i\,.8510,\ 4.244 \rangle$	56.58
Centered Form	$\langle .4768 - i\,.8510,\ 3.816 \rangle$	45.75

The graphical output (Fig. 1) shows that the centered form generates the smallest circle.

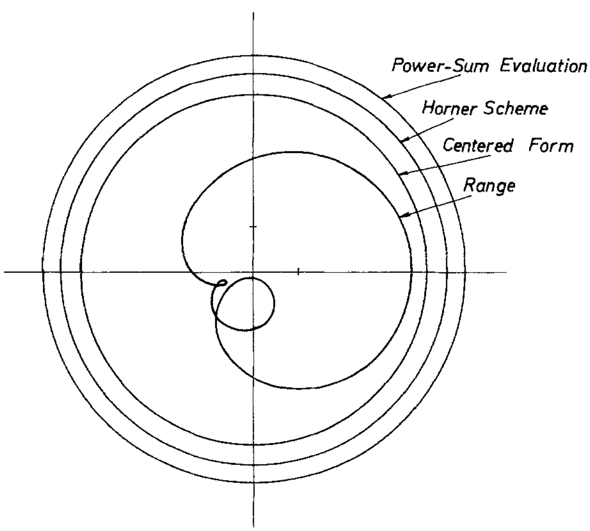

Fig. 1

Example 2: Here we choose the polynomial

$$\begin{aligned} p(z) = &(.1062 + i\,.9162) + (.3921 + i\,.2056)\,z \\ &+ (.5912 + i\,.4821)\,z^2 + (.2174 + i\,.8121)\,z^3 \\ &+ (.3821 + i\,.3011)\,z^4 + (.5462 - i\,.7011)\,z^5 \\ &+ (.3216 + i\,.6731)\,z^6 + (.1005 + i\,.5001)\,z^7 \end{aligned}$$

to be evaluated over the disk $\langle .1203 + i.2011,\ .7736 \rangle$. The numerical results are shown in the following table:

Table 2

Algorithm	Resulting Circle	Area of Resulting Circle
Power Sum	$\langle .07 + i\,1.024, 4.671 \rangle$	68.54
Hornerscheme	$\langle 0.07 + i\,1.024, 3.86 \rangle$	46.81
Centered Form	$\langle 0.07 + i\,1.024, 2.55 \rangle$	20.43

From Fig. 2 it is clear that the centered form generates the best result.

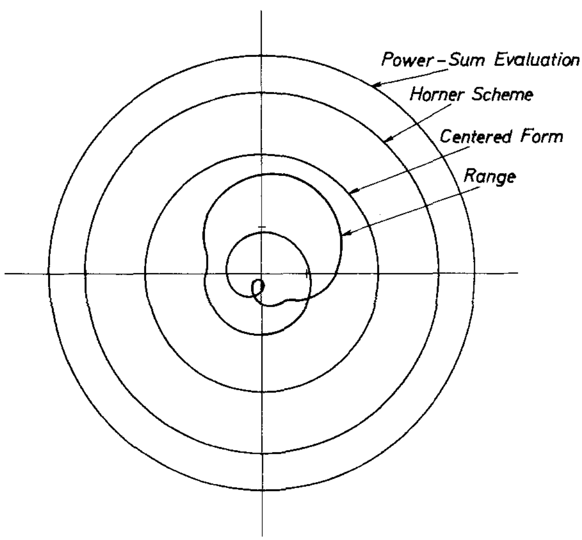

Fig. 2

Example 3: In this final example, we choose the polynomial

$$\begin{aligned} p(z) = &(.5005 - i\,.9003) + (.3056 + i\,.9021)\,z \\ &+ (.4056 + i\,.6023)\,z^2 + (.2978 + i\,.8271)\,z^3 \\ &+ (.4821 + i\,.7921)\,z^4 + (.9026 - i\,.4023)\,z^5 \\ &+ (.2185 - i\,.5036)\,z^6 \end{aligned}$$

to be evaluated over the domain $\langle -.2615 - i.4013,\ 5572 \rangle$.

The results are

Table 3

Algorithm	Resulting Circle	Area of Resulting Circle
Power Sum	$\langle .6839 - i\,1.174, 4.874\rangle$	74.63
Hornerscheme	$\langle .6839 - i\,1.174, 2.419\rangle$	18.38
Centered Form	$\langle .6839 - i\,1.174, 0.9006\rangle$	2.545

The graphical output is shown in Fig. 3. It is easy to see that a great improvement is achieved by the centered form.

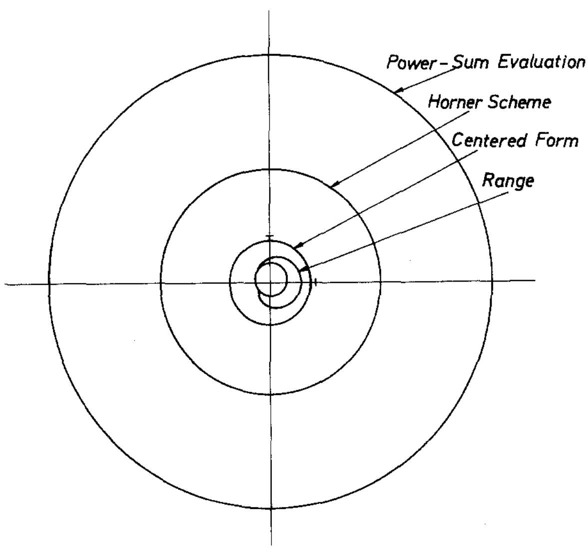

Fig. 3

5. Conclusion

In [7] it was shown that the circular complex centered form for a polynomial had quadratic convergence towards the range of the polynomial.

Several numerical examples showed that this form gave good results, also always being better experimentally than the power-sum evaluation.

In this note this fact is shown to be always true and that the circular complex centered form is always better than the Hornerscheme as well as the power-sum evaluation.

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